ASYMPTOTIC SOLUTION OF A CLASS OF INTEGRAL EQUATIONS ENCOUNTERED IN THE INVESTIGATION OF MIXED PROBLEMS OF THE MATHEMATICAL PHYSICS FOR REGIONS WITH CYLINDRICAL BOUNDARIES

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We consider a special class of integral equations of the first kind with an irregular difference kernel of complex structure, dependent on a dimensionless parameter λ. We construct an asymptotic solution of this equation for small values of λ. Use of the Wiener-Hopf method and introduction of a class of special approximations to the Fourier kernel transform, allows us to perform an approximate factorization and thus bring the given problem to the numerical stage.

The results obtained are used to investigate axisymmetric problems of the interaction between a stiff tire and the surface of an infinite elastic cylinder as well as the interaction between a stiff bushing and the surface of an infinite cylindrical cavity in an elastic space. Solutions are obtained in the form of fairly simple expressions, which coincide asymptotically with the corresponding solutions of [1]. Thus the method given in [1] and the method derived in the present paper, make possible the complete investigation of the given class of integral equations over the whole range of values of λ.

1. Structure of the kernel of the integral equation and of its solution. Let us consider an integral equation of the form

$$\int_{-1}^{1} \varphi(\tau) K\left(\frac{\tau - x}{\lambda}\right) d\tau = \pi f(x), \qquad |x| \leqslant 1 \tag{1.1}$$

$$K(t) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{L(u)}{u} e^{-iut} du \qquad \left(t = \frac{\tau - x}{\lambda}\right) \tag{1.2}$$

The function $L(u)u^{-1}$ is assumed even, real and regular on the real axis $-\infty < u < \infty$; moreover we assume

$$L(u)u^{-1} = A + O(u^2) \quad \text{as} \quad u \to 0 \tag{1.3}$$

$$L(u)u^{-1} = |u|^{-1}[1 + c_1|u|^{-1} + c_2u^{-2} + O(|u|^{-3})], \quad |u| \to \infty \tag{1.4}$$

Using the above properties of $L(u)u^{-1}$, we can show the structure of the kernel $K(t)$. It is

$$K(t) = -\ln|t| + a_{20}|t| + a_{30} + F(t), \qquad 0 \leqslant t < \infty \tag{1.5}$$

$$F(t)=\frac{c_2t^2}{2}\ln|t|-\frac{3c_2t^2}{4}+\int_0^\infty\frac{[uL(u)-u-c_1-c_2u^{-1}](\cos ut-1)-{}^1/_2c_2ue^{-u}t^2}{u^2}\,du$$

$$a_{30}=\int_0^\infty\frac{L(u)-1+e^{-u}}{u}\,du,\qquad a_{20}=-\frac{1}{2}\pi c_1 \tag{1.6}$$

Here we used the formulas (2.6) of [1]. Next we shall prove the following lemma.

L e m m a 1.1. The function $F(t)$ of the form (1.6) belongs to the class of functions

$$H_1^{\alpha}(-2/\lambda,\ 2/\lambda),\qquad 1-\varepsilon<\alpha<1,\quad \varepsilon>0$$

for all $0\leq|t|<\infty$ (definition of the class of the functions $H_n^{\alpha}(-\beta,\beta)$ is given in [1]).

P r o o f. Let us find the second derivative of $F(t)$ given by (1.6)

$$F''(t)=c_2\ln|t|-\int_0^\infty\left\{\left[uL(u)-u-c_1-\frac{c_2}{u}\right]\cos ut+\frac{c_2}{u}e^{-u}\right\}du \tag{1.7}$$

We note that the properties (1.3) and (1.4) of $L(u)u^{-1}$ imply that the integral in (1.7) converges uniformly for all $0\leq|t|<\infty$, and the proof follows from this.

Let us now turn our attention to the structure of the solution of (1.1). We shall require the following lemma.

L e m m a 1.2. If $\gamma(t)\in H_m^{\alpha}(-1,1)$ where $\alpha>½$ when $1-\varepsilon<|t|\leq 1$ and $\alpha>0$ when $0\leq|t|\leq 1-\varepsilon$, $\varepsilon>0$, then

$$I_1(x)=\int_{-1}^{1}\frac{\gamma(t)\,dt}{\sqrt{1-t^2}\,(t-x)}\in C_m(-1,1),\qquad |x|\leqslant 1 \tag{1.8}$$

P r o o f. Using the well known relation

$$\int_{-1}^{1}\frac{dt}{(t-x)\sqrt{1-t^2}}=0$$

we can write the integral (1.8) as

$$I_1(x)=\int_{-1}^{1}\frac{\gamma(t)-\gamma(x)}{(t-x)\sqrt{1-t^2}}\,dt \tag{1.9}$$

which, differentiated formally m times with respect to x, yields

$$I_1^{(m)}(x)=m!\int_{-1}^{1}\frac{\gamma(t)-\gamma(x)-(t-x)/1!\,\gamma'(x)-\ldots-(t-x)^m/m!\,\gamma(x)^{(m)}}{(t-x)^{m+1}\sqrt{1-t^2}}\,dt \tag{1.10}$$

Using now the identities [2]

$$\gamma(t)-\gamma(x)-\frac{(t-x)}{1!}\gamma'(x)-\ldots-\frac{(t-x)^{m-1}}{(m-1)!}\gamma(x)^{(m-1)}=\frac{1}{(m-1)!}\int_x^t(t-\tau)^{m-1}\gamma^{(m)}(\tau)\,d\tau$$

$$\frac{(t-x)^m}{m}=\int_x^t(t-\tau)^{m-1}\,d\tau$$

we can obtain the following estimate for the numerator of the integrand in (1.10)

$$|\gamma(t)-\gamma(x)-(t-x)-1!\,\gamma'(x)-\ldots-(t-x)^m-m!\,\gamma^{(m)}(x)|= \tag{1.11}$$

$$=\left|\frac{(t-x)^m}{m!}[\gamma(t)^{(m)}-\gamma(x)^{(m)}]-\frac{1}{(m-1)!}\int_x^t(t-\tau)^{m-1}[\gamma^{(m)}(t)-\gamma^{(m)}(\tau)]\,d\tau\right|\leqslant$$

$$\leqslant\frac{A}{m!}(t-x)^{m+\alpha}+\frac{A}{(m-1)!}\int_x^t(t-\tau)^{m+\alpha-1}\,d\tau=B\,|t-x|^{m+\alpha}$$

In the derivation of (1.11) we assumed, without any loss of generality, that $t \geq x$.

Next we shall prove that $I_1^{(m)}(x) \in C(-1, 1)$. For this, it will be sufficient to show that the integral (1.10) converges uniformly for all $x \in [-1,1]$. This will also justify the differentiation under the integral sign in (1.9). Uniform convergence of the integral (1.10) can be shown in a fairly straightforward manner, using the estimate (1.11), and this proves the lemma.

Corollary 1.1. If $\gamma(t) \in H_m^\alpha(-1, 1)$, $\alpha > 0$, then

$$I_2(x) = \int_{-1}^{1} \frac{\gamma(t)\sqrt{1-t^2}}{t-x}\, dt \in C_m(-1, 1), \qquad |x| \leqslant 1$$

(formulation of this corollary was given in [3]).

Corollary 1.2. If $\gamma(t) \in H_m^\alpha(-1+\varepsilon, 1)$, $\alpha > 0$, $\varepsilon > 0$ and $\gamma(t) \in H_m^\beta(-1, -1+\varepsilon)$, $\beta > 1/2$, then

$$I_3(x) = \int_{-1}^{1} \frac{\gamma(t)}{t-x}\left(\frac{1-t}{1+t}\right)^{1/2} dt \in C_m(-1, 1), \qquad |x| \leqslant 1$$

An analogous corollary exists for the integral

$$I_4(x) = \int_{-1}^{1} \frac{\gamma(t)}{t-x}\left(\frac{1+t}{1-t}\right)^{1/2} dt \tag{1.12}$$

We shall further assume that the function $f(x)$ appearing in the right-hand side of the integral Eq. (1.1) belongs, at least, to $H_1^\beta(-1, 1)$, $0 < \beta \leq 1$. Then the following theorem (which was formulated in [1] but had some errors, now removed, in its proof), exists.

Theorem 1.1. If a solution of the integral Eq. (1.1) exists in the class of functions $L_p(-1, 1)$, $1+\delta > p > 1$, $\delta > 0$, then for any value of $\lambda \in (0, \infty)$ it will have the form $\phi(x) = (1-x^2)^{-1/2}\omega_1(x)$ where $\omega_1(x) \in C(-1, 1)$.

Proof. We shall represent the integral Eq. (1.1) with the kernel given by (1.5) in the form of an equivalent integral equation of the second kind

$$\varphi(x) = \frac{1}{\pi\sqrt{1-x^2}}\Bigg\{P - \int_{-1}^{1} \frac{f'(t)\sqrt{1-t^2}}{t-x}\, dt + \frac{1}{\pi\lambda}\int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x}\, dt \int_{-1}^{1}\left[a_{20}\,\mathrm{sgn}(t-y) + F'\left(\frac{t-y}{\lambda}\right)\right]\varphi(y)\, dy\Bigg\}, \qquad P = \int_{-1}^{1}\varphi(y)\, dy \tag{1.13}$$

with the condition

$$P = \frac{1}{\ln 2\lambda + a_{30}}\left\{\int_{-1}^{1}\frac{f(t)\,dt}{\sqrt{1-t^2}} - \frac{1}{\pi}\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}}\int_{-1}^{1}\left[a_{20}\frac{|t-y|}{\lambda} + F\left(\frac{t-y}{\lambda}\right)\right]\varphi(y)\, dy\right\} \tag{1.14}$$

Assumption of the Theorem implies that $\phi(x) \in L_p(-1, 1)$, then even more so $\phi(x) \in L(-1, 1)$. Then $P < \infty$, and using the result of Lemma 1.1 we can prove, that

$$J(t) = \int_{-1}^{1} F'\left(\frac{t-y}{\lambda}\right)\varphi(y)\, dy \in H_0^\alpha(-1, 1)$$

Now, the properties of the function $f(x)$ given above and the Corollary 1.1, bring us to conclusion that

$$-\int_{-1}^{1}\frac{f'(t)\sqrt{1-t^2}}{t-x}\, dt + \frac{1}{\pi\lambda}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}\, dt\int_{-1}^{1} F'\left(\frac{t-y}{\lambda}\right)\varphi(y)\, dy \in C(-1, 1)$$

To prove the theorem it remains to show, that

$$I(x) = \int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x} dt \int_{-1}^{1} \operatorname{sgn}(t-y)\,\varphi(y)\,dy \in C(-1, 1) \tag{1.15}$$

Let us write the inner integral of (1.15) as

$$N(t) = 2\int_{-1}^{t} \varphi(y)\,dy - P$$

Then, using the Hölder inequality we can prove that

$$N(t) \in H_0^{1/q}(-1, 1),\ 1/q + 1/p = 1$$

which, together with the Corollary 1.1, proves the validity of (1.15) and this completes the proof.

Next we shall see, whether a solution of (1.1) bounded at one or at both ends $x = \pm 1$ of the line of contact, can be obtained.

Theorem 1.2. If the function $f(x) \in H_1^{\alpha}(-1+\varepsilon, 1)$, $\alpha > 0$ and $f(x) \in H_1^{\beta}(-1, -1+\varepsilon)$, $\beta > 1/2$, a solution of (1.1) exists in the class of functions $L_p(-1, 1)$, $1+\delta > p > 1$ and is bounded in the ε-neighborhood of the point $x = -1$, then for any $\lambda \in (0, \infty)$ this solution will have the form $\phi(x) = (1+x)^{1/2}(1-x)^{-1/2}\omega_2(x)$, where $\omega_2(x) \in C(-1, 1)$ and where the following relation holds

$$P = -\int_{-1}^{1} f'(t)\left(\frac{1+t}{1-t}\right)^{1/2} dt -$$

$$-\frac{1}{\pi\lambda}\int_{-1}^{1}\left(\frac{1+t}{1-t}\right)^{1/2} dt \int_{-1}^{1} \varphi(\tau)\left[a_{20}\operatorname{sgn}(\tau-t) + F'\left(\frac{\tau-t}{\lambda}\right)\right] d\tau \tag{1.16}$$

The proof is analogous to that of the Theorem 1.1. We must, however, use the Corollary 1.2 of the Lemma 1.2 and the integral Eq.

$$\varphi(x) = -\frac{1}{\pi}\left(\frac{1+x}{1-x}\right)^{1/2}\left\{\frac{1}{\pi\lambda}\int_{-1}^{1}\left(\frac{1-t}{1+t}\right)^{1/2}\frac{dt}{t-x}\int_{-1}^{1}\varphi(\tau)\left[a_{20}\operatorname{sgn}(\tau-t) + \right.\right.$$

$$\left.\left. + F'\left(\frac{\tau-t}{\lambda}\right)\right] d\tau + \int_{-1}^{1}\frac{f'(t)}{t-x}\left(\frac{1-t}{1+t}\right)^{1/2} dt\right\} \tag{1.17}$$

which is equivalent to (1.1) under the conditions (1.14) and (1.16).

An analogous theorem exists when the solution of (1.1) is assumed bounded in the ε-neighborhood of the point $x = 1$.

Theorem 1.3. If the function $f(x) \in H_1^{\alpha}(-1+\varepsilon, 1-\varepsilon)$, $\alpha > 0$, $\varepsilon > 0$; $f(x) \in H_1^{\beta}(-1, -1+\varepsilon)$, $\beta > 1/2$; $f(x) \in H_1^{\gamma}(1-\varepsilon, 1)$, $\gamma > 1/2$, a solution of (1.1) exists in the class of functions $L_p(-1, 1)$ and is bounded in the ε-neighborhood of the points $x = \pm 1$, then for any $\lambda \in (0, \infty)$ it has the form $\phi(x) = (1-x^2)^{1/2}\omega_3(x)$. Here $\omega_3 \in C(-1, 1)$ and the relations

$$P = -\int_{-1}^{1}\frac{tf'(t)\,dt}{\sqrt{1-t^2}} - \frac{1}{\pi\lambda}\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}}\int_{-1}^{1}\varphi(\tau)\left[a_{20}\operatorname{sgn}(\tau-t) + F'\left(\frac{\tau-t}{\lambda}\right)\right] d\tau$$

$$0 = \int_{-1}^{1}\frac{f'(t)\,dt}{\sqrt{1-t^2}} + \frac{1}{\pi\lambda}\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}}\int_{-1}^{1}\varphi(\tau)\left[a_{20}\operatorname{sgn}(\tau-t) + F'\left(\frac{\tau-t}{\lambda}\right)\right] d\tau$$

hold. The proof is again analogous to that of the Theorem 1.1. Lemma 1.2 must however, be used together with the integral Eq.

$$\varphi(x)=-\frac{1}{\pi}\sqrt{1-x^2}\left\{\frac{1}{\pi\lambda}\int_{-1}^{1}\frac{dt}{(t-x)\sqrt{1-t^2}}\int_{-1}^{1}\varphi(\tau)\left[a_{20}\operatorname{sgn}(\tau-t)+\right.\right.$$

$$\left.\left.+F'\left(\frac{\tau-t}{\lambda}\right)\right]d\tau+\int_{-1}^{1}\frac{f'(t)\,dt}{\sqrt{1-t^2}(t-x)}\right\}$$

which is equivalent to (1.1) under the conditions (1.14) and (1.18).

2. Method of small λ. Stability of the solution of the integral equation (1.1). Construction of the class of possible approximations to its kernel. Since under the assumptions made previously the function $f(x)$ can be expanded into a Fourier series, we find that the linearity of the integral Eq. (1.1) implies that it is sufficient to obtain a solution to an integral equation of the following particular type

$$\int_{-1}^{1}\varphi_\eta(\tau)K\left(\frac{\tau-x}{\lambda}\right)d\tau=\pi\gamma e^{i\eta x},\qquad |x|\leqslant 1 \tag{2.1}$$

We propose to obtain an approximate solution of (2.1) for small values of λ. It was shown in [3] that the zeroth term of the asymptotic form of the solution for small λ can be represented as

$$\varphi_\eta(x)=\psi_+\left(\frac{1-x}{\lambda}\right)+\psi_-\left(\frac{1+x}{\lambda}\right)-\psi_0\left(\frac{x}{\lambda}\right) \tag{2.2}$$

where the functions $\psi_\pm(t)$ and $\psi_0(t)$ are given by the following integral Eqs.

$$\int_{0}^{\infty}\psi_\pm(\tau)K(t-\tau)\,d\tau=\pi\gamma_\pm e^{\mp i\beta t},\qquad 0\leqslant t<\infty \tag{2.3}$$

$$\int_{-\infty}^{\infty}\psi_0(\tau)K(t-\tau)\,d\tau=\pi\gamma_0 e^{i\beta t},\qquad |t|<\infty \tag{2.4}$$

$$\gamma_\pm=\gamma\lambda^{-1}e^{\pm i\eta},\qquad \gamma_0=\gamma\lambda^{-1},\qquad \beta=\lambda\eta$$

Solution of the integral Eq. (2.4) can be obtained using the convolution theorem for the integral Fourier transform, and has the form

$$\psi_0(t)=\gamma_0\,\beta L^{-1}(\beta)\,e^{i\beta t} \tag{2.5}$$

Solution of (2.3) can be obtained using the Wiener-Hopf method [4]. Koiter has shown in [5] that in order to bring the solution to the form suitable for computation, it is expedient to resort to approximate factorization, and he proposed the following approximation to the Fourier transform of $L(u)u^{-1}$ appearing in the kernel $K(t)$

$$\frac{L^*(u)}{u}=\frac{1}{\sqrt{u^2+A^{-2}}}\frac{P_1(u)}{P_2(u)}\qquad (P_1(0)=P_2(0)=B=\text{const}) \tag{2.6}$$

where $P_1(u)$ and $P_2(u)$ are even polynomials of equal degree. Authors of [6] give a more general approximation of the form

$$\frac{L^*(u)}{u}=\frac{\sqrt{u^2+B^2}}{u^2+C^2}\prod_{n=1}^{N}\frac{(u^2+D_n^2)}{(u^2+E_n^2)}\qquad\left(\frac{B}{C^2}\prod_{n=1}^{N}\frac{D_n^2}{E_n^2}=A\right) \tag{2.7}$$

which gives a high accuracy at small number N.

In our opinion, the following, easily factorizable approximation also merits attention

$$\frac{L^*(u)}{u}=\frac{\sqrt{u^2+C^2}}{\varkappa^2+\varkappa\sqrt{(C-iu)(D-iu)}+\varkappa\sqrt{(C+iu)(D+iu)}+\sqrt{(C^2+u^2)(D^2+u^2)}}\times$$

$$\times \prod_{n=1}^{N} \frac{(u^2 + D_n^2)}{(u^2 + E_n^2)} \frac{C}{(\varkappa + \sqrt{CD})^2} \prod_{n=1}^{N} \frac{D_n^2}{E_n^2} = A, \qquad \varkappa > 0 \tag{2.8}$$

In the number of cases it yields a more accurate structural representation of $L(u)\,u^{-1}$, thus raising the accuracy of the solution.

It is easily seen that all approximations given by (2.6) to (2.8) satisfy the property (1.3) of $L(u)\,u^{-1}$; but when $|u| \to \infty$, their asymptotic form is

$$L^*(u)\ u^{-1} = |u|^{-1}\ [1 + c_2{}^* u^{-2} + O(u^{-4})] \tag{2.9}$$

This yields the following asymptotic representation for the kernel

$$K^*(t) = -\ln |t| + a_{30}{}^* + F^*(t) \quad (0 \leqslant t < \infty)$$

$$F^*(t) = O(t^2 \ln |t|) \quad \text{при } t \to 0 \tag{2.10}$$

Let us compare Formulas (1.4) to (1.6) with (2.9) and (2.10). We can easily see that the absence of the term $|u|^{-1}$ within the square bracket in (2.9) leads to the absence of the term $|t|$ in (2.10). Thus an inaccuracy occurring in the behavior of the approximating function $L^*(u)H$ as $|u| \to \infty$ would lead to conclusion that the function $K^*(t) + \ln|t|$ is smoother than $K(t) + \ln|t|$.

We shall show, how this may influence the accuracy of the asymptotic solution of (1.1) at small values of λ. We shall consider (1.1), (1.5) and an integral Eq. of the form

$$\int_{-1}^{1} \varphi_1(\tau) \left[-\ln \frac{|\tau - x|}{\lambda} + b_{20} \frac{|\tau - x|}{\lambda} + b_{30} + G\left(\frac{\tau - x}{\lambda}\right)\right] d\tau = \pi g(x), \quad |x| \leqslant 1 \tag{2.11}$$

We shall assume that, similarly to (1.1) and (1.5), we have

$$g(x) \in H_1{}^\beta(-1, 1), \quad \beta > 0; \qquad G(t) \in H_1{}^\alpha(-2/\lambda, 2/\lambda)$$
$$1 - \varepsilon < \alpha < 1, \ \varepsilon > 0$$

The integral Eq. (2.11) shall be called perturbed with respect to (1.1) and (1.5), if the following conditions hold

$$|a_{20} - b_{20}| \leqslant \varepsilon, \qquad |a_{30} - b_{30}| \leqslant \varepsilon$$
$$\|f(x) - g(x)\|_{H_1{}^\alpha(-1,1)} \leqslant \varepsilon, \qquad \|F(t) - G(t)\|_{H_1{}^\alpha(-2/\lambda, 2/\lambda)} \leqslant \varepsilon \tag{2.12}$$

The norm on the space $H_1{}^\alpha(-\beta, \beta)$ will be given by

$$\|f(x)\|_{H_1{}^\alpha(-\beta,\beta)} = \max|f(x)| + \max|f'(x)| +$$
$$+ \sup \frac{|f'(t) - f'(x)|}{|t - x|^\alpha} \quad (x, t) \in [-\beta, \beta] \tag{2.13}$$

It can easily be shown that $H_1{}^\alpha(-\beta, \beta)$ is a complete, linear, normed space. From the definition of the norm on $H_1{}^\alpha(-\beta, \beta)$ it follows that, if

$$\|f(x)\|_{H_1{}^\alpha(-1,1)} \leqslant \varepsilon,$$

then

$$\|f(x)\|_{C(-1,1)} \leqslant \varepsilon, \quad \|f'(x)\|_{C(-1,1)} \leqslant \varepsilon \leqslant |f'(t) - f'(x)| \leqslant \varepsilon |t - x|^\alpha \tag{2.14}$$

We shall now prove the following theorem.

Theorem 2.1. If solutions exist in the class of functions $L_p(-1, 1)$, $1 + \delta > p > 1$, $\delta > 0$ for both, the integral Eq. (1.1) and (1.6) and another integral equation perturbed with respect to the former, then the estimate

$$\|\varphi(x) - \varphi_1(x)\|_{C(-1,1)} \leqslant A\varepsilon (1 - x^2)^{-1/2}, \qquad A = \text{const} \tag{2.15}$$

holds for any $\lambda \in (0, \infty)$.

Proof. Let us represent the integral Eq. (2.11) by an equivalent integral equation of the second kind analogous to (1.13) and (1.14). The difference between the obtained equation and (1.13) will be

$$\psi(x)+\frac{1}{\pi^2(\ln 2\lambda+b_{30})}\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}}\int_{-1}^{1}\left[b_{20}\frac{|t-y|}{\lambda}+G\left(\frac{t-y}{\lambda}\right)\right]\frac{\psi(y)\,dy}{\sqrt{1-y^2}}-$$

$$-\frac{1}{\pi^2\lambda}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}\,dt\int_{-1}^{1}\left[b_{20}\operatorname{sgn}(t-y)+G'\left(\frac{t-y}{\lambda}\right)\right]\frac{\psi(y)\,dy}{\sqrt{1-y^2}}=\gamma(x)$$

$$\gamma(x)=\frac{1}{\pi(\ln 2\lambda+b_{30})}\left\{-c_{30}P+\int_{-1}^{1}\frac{f_1(t)\,dt}{\sqrt{1-t^2}}-\frac{1}{\pi}\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}}\int_{-1}^{1}\left[c_{20}\frac{|t-x|}{\lambda}+\right.\right.$$

$$\left.\left.+F_1\left(\frac{t-x}{\lambda}\right)\right]\varphi(x)\,dx\right\}+\frac{1}{\pi}\int_{-1}^{1}\frac{f_1'(t)\sqrt{1-t^2}}{t-x}\,dt-$$

$$-\frac{1}{\pi^2\lambda}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}\,dt\int_{-1}^{1}\left[c_{20}\operatorname{sgn}(t-y)+F_1'\left(\frac{t-y}{\lambda}\right)\right]\varphi(y)\,dy \qquad (2.16)$$

where the term $P-P_1$ is replaced, in accordance with (1.14), by

$$P-P_1=\left\{-c_{30}P+\int_{-1}^{1}\frac{f_1'(t)\,dt}{\sqrt{1-t^2}}-\frac{1}{\pi}\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}}\int_{-1}^{1}\left[b_{20}\frac{|t-y|}{\lambda}\times G\left(\frac{t-y}{\lambda}\right)\right]\right.$$

$$\left.\frac{\psi(y)}{\sqrt{1-y^2}}\,dy-\frac{1}{\pi}\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}}\int_{-1}^{1}\left[c_{20}\frac{|t-y|}{\lambda}+F_1\left(\frac{t-y}{\lambda}\right)\right]\varphi(y)\,dy\right\}\frac{1}{\ln 2\lambda+b_{30}} \qquad (2.17)$$

The following notation is also used in (2.16) and (2.17)

$$(1-x^2)^{-1/2}\,\psi(x)=\varphi(x)-\varphi_1(x),\ c_{20}=a_{20}-b_{20},\qquad c_{30}=a_{30}-b_{30}$$

$$f_1(x)=f(x)-g(x),\qquad F_1(t)=F(t)-G(t) \qquad (2.18)$$

We shall now investigate the integral operator appearing in the left hand side of (2.16). From the Theorem 1.1 it follows that $\psi(x)\in C(-1,1)$, therefore taking into account the fact that $G(t)\in H_1^{\alpha}(-2/\lambda,2/\lambda)$, we easily arrive at the conclusion that the following function of y exists in $L(-1,1)$

$$\{b_{20}\operatorname{sgn}(t-y)/\lambda+G'[(t-y)/\lambda]\}(1-y^2)^{-1/2}\,\psi(y)\in L(-1,1)$$

Then the Lemma (see e.g. [7], ch. 1, Section 7) will imply the possibility of changing the order of integration in the third term of the left-hand side of (2.16) and even more so in the second term. Consequently we can rewrite (2.16) as

$$\psi(x)+\int_{-1}^{1}\psi(y)M(y,x)\,dy=\gamma(x),\qquad |x|\leqslant 1 \qquad (2.19)$$

where

$$M(y,x)=-\frac{1}{\pi^2\sqrt{1-y^2}}\left\{\frac{1}{\lambda}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}\left[b_{20}\operatorname{sgn}(t-y)+G'\left(\frac{t-y}{\lambda}\right)\right]dt-\right.$$

$$\left.-\frac{1}{\ln 2\lambda+b_{30}}\int_{-1}^{1}\left[b_{20}\frac{|t-y|}{\lambda}+G\left(\frac{t-y}{\lambda}\right)\right]\frac{dt}{\sqrt{1-t^2}}\right\} \qquad (2.20)$$

Let us investigate the properties of the kernel (2.20) of the integral Eq. (2.19). Obviously, the second integral in (2.20) is a function of y, and therefore belongs to $H_1^{\alpha}(-1,1)$, $1-\varepsilon<\alpha<1$. Further, since $G'(t)\in H_0^{\alpha}(-2/\lambda,2/\lambda)$, we find that by the Corollary 1.1, the integral

$$\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}G'\left(\frac{t-y}{\lambda}\right)dt\in C(-1,1)$$

as a function of x and is, at least, bounded in y. The latter can easily be shown. Finally let us consider the integral

$$\int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x} \operatorname{sgn}(t-y)\,dt = 2x \arcsin y - 2\sqrt{1-y^2} +$$
$$+ \sqrt{1-x^2} \ln \frac{\sqrt{(1-x^2)(1-y^2)} + 1 - xy}{|y-x|} \qquad (x, y) \in [-1, 1]$$

We easily see that its only singularity is logarithmic, at $x = y \neq \pm 1$.

Thus the kernel $M(y, x)$ has a singularity of the type $(1-y^2)^{-1/2}$ at $y = \pm 1$ and a logarithmic singularity at $x = y \neq \pm 1$. At other values of x and y it is, at least, bounded.

Let us now perform the variable substitution $y = \sin \tau$, $x = \sin \theta$ in (2.19). This will yield an integral equation whose kernel will retain only the logarithmic singularity and which, consequently, is the Fredholm equation [8]. Then, by the Lemma of [9] and under the assumption that (2.19) has a unique solution, we have

$$|\psi(\sin\theta)| = |\psi(x)| \leqslant B\,|\gamma(x)|, \quad B = \text{const} \tag{2.21}$$

Let us now estimate $\gamma(x)$. Taking into account (2.12) we easily find that $|\gamma(x)| \leq \varepsilon B_1$. Inserting this into (2.21), we obtain (2.15). We also note that

$$|P - P_1| \leqslant \varepsilon D_1, \quad D_1 = \text{const} \tag{2.22}$$

follows from (2.15), and this completes the proof.

The following theorems can also be proved in the similar manner.

T h e o r e m 2.2. If solutions exist in the class of functions $L_p(-1, 1)$, $1+\delta > p > 1$ for both, the integral Eqs. (1.1) and (1.5) and another integral equation perturbed with respect to the former, if these solutions are bounded in the ε-neighborhood of the point $x = -1$ and if the relation (1.16) holds for each of them, then the following estimate

$$\|\varphi(x) - \varphi_1(x)\|_{C(-1,1)} \leqslant \varepsilon A\,(1-x)^{-1/2}, \qquad A = \text{const} \tag{2.23}$$

holds for any $\lambda \in (0, \infty)$.

If we assume that both of the above solutions are bounded in the ε-neighborhood of the point $x = 1$, then we have the following analogous theorem.

T h e o r e m 2.3. If a solution of the integral Eqs. (1.1) and (1.5) and a solution of another integral equation perturbed with respect to the former, both exist in the class of functions $L_p(-1, 1)$, $1+\delta > p > 1$, are bounded in the ε-neighborhood of the points $x = \pm 1$ and if relations (1.18) hold for both solutions, then the estimate

$$\|\varphi(x) - \varphi_1(x)\|_{C(-1,1)} \leqslant \varepsilon A, \qquad A = \text{const} \tag{2.24}$$

holds for any $\lambda \in (0, \infty)$.

Thus, when the conditions (2.12) hold, the Theorems 2.1 to 2.3 guarantee that the approximate solution of (1.1) with the approximated kernel will deviate little from the exact solution. It can, however, be easily seen that the function $K^*(t) + \ln|t|$ of the form of (2.10) and obtained from one of the approximate expressions (2.6) to (2.8) does not, in general, satisfy the first condition of (2.12). Therefore, when we use the approximations (2.6) to (2.8) to obtain an approximate solution to (1.1) for small λ, we cannot be absolutely certain that the solution obtained will deviate from the exact solution by only a small amount. Indeed, numerical analysis of examples based on the approximation (2.7) for small λ, gave a poor agreement with results known to be practically exact and obtained by other methods.

Thus the behavior of the function $L(u)u^{-1}$ as $|u| \to \infty$ and the theorems given above, imply the necessity of constructing approximations other than (2.6) to (2.8).

At this stage we note that we can always represent $L(u)u^{-1}$ as a sum of two functions

$$\begin{aligned} L(u)\,u^{-1} &= L_1(u)\,u^{-1} + L_2(u)\,u^{-1} \\ L_1(u)\,u^{-1} &= 0.5\,|u|^{-1}\,[L(|u|) + L(-|u|)] \\ L_2(u)\,u^{-1} &= 0.5\,|u|^{-1}\,[L(|u|) - L(-|u|)] \end{aligned} \tag{2.25}$$

and, when $|u| \to \infty$, we obviously have

$$L_1(u)\,u^{-1} = |u|^{-1}\,[1 + c_2u^{-2} + O(u^{-4})]$$
$$L_2(u)\,u^{-1} = |u|^{-1}\,[c_1|u|^{-1} + c_3|u|^{-3} + O(|u|^{-5})]$$

This suggests at once, that $L_1(u)\,u^{-1}$ and $L_2(u)\,u^{-1}$ could be approximated separately. One of the approximations given by (2.6) to (2.8) will, obviously, be suitable for $L_1(u)\,u^{-1}$, while

$$\frac{L_2^*(u)}{u} = \frac{c_1}{(u^2+c^2)}\prod_{n=1}^{N}\frac{(u^2+d_n^2)}{(u^2+l_n^2)} \tag{2.26}$$

should approximate the other function fairly well.

We can do it in a slightly different way, namely by approximating the whole function $L(u)\,u^{-1}$ first, using one of the expressions (2.6) to (2.8). This will not be very accurate even at large N, since, as we have already said, the functions (2.6) to (2.8) do not describe the behavior of $L(u)\,u^{-1}$ completely. Next step will consist of approximating the difference $L(u)\,u^{-1} - L^*(u)\,u^{-1}$ by means of an expression of the form (2.26). Such an approach appears to be suitable when the Wiener-Hopf integral Eq. (2.3) is solved by the method of successive approximations (see Section 3) and this method converges at the rate, which is inversely proportional to the value of

$$\max|L(u)\,u^{-1} - L^*(u)\,u^{-1}|$$

Finally, we shall mention yet another method of approximating the function $L(u)\,u^{-1}$: first we approximate $L(u)\,u^{-1}$ using one of the expressions (2.6) to (2.8), and we follow it by approximating the ratio $L(u)/L^*(u)$ with a function of the form $\exp[c_1 M(u)]$ where $M(u)$ is a function analogous to (2.6) to (2.8). We easily see that the approximation constructed in this manner will satisfy the conditions (1.3) and (1.4).

3. Solution of the Wiener-Hopf integral equation. Without going into great detail, we shall represent the integral Eq. (2.3) by an equivalent functional equation [4]

$$\Phi_+(\alpha)\;L(\alpha)\,\alpha^{-1} = F_+(\alpha) + E_-(\alpha) \tag{3.1}$$

where

$$\Phi_+(\alpha) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \psi_\pm(\tau)\,e^{i\alpha\tau}d\tau, \qquad F_+(\alpha) = -\frac{\gamma_\pm}{i\sqrt{2\pi}\,(\alpha\pm\beta)}$$

$$E_-(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 e_\pm(\tau)\,e^{i\alpha\tau}d\tau, \qquad e_\pm(x) = 2\pi\int_0^\infty \psi_\pm K(x-\tau)\,d\tau \tag{3.2}$$
$$0>x>-\infty$$

Let us now consider the approximate solution of (3.1). Writing $L(u)\,u^{-1}$ in the form of (2.25) and taking (2.7) and (2.28) into account, we obtain

$$\Phi_+(\alpha)\;L_1^*(\alpha)\,\alpha^{-1} + \varepsilon\,\Phi_+(\alpha)\;L_2^*(\alpha)\,\alpha^{-1} = F_+(\alpha) + E_-(\alpha), \quad \varepsilon=1 \tag{3.2}$$

where the parameter ε is brought in for convenience. Taking into account the fact that $\max|L_2^*(u)\,u^{-1}|$ is small compared with $L_1^*(u)\,u^{-1}$ (we can show this by approximating the function $L(u)\,u^{-1}$ as shown in Section 2), we shall seek the solution of (3.2) in the form (see e.g. [4], Sections 4 and 5)

$$\Phi_+(\alpha) = \Phi_+^{(0)} + \varepsilon\Phi_+^{(1)} + \varepsilon^2\Phi_+^{(2)} + \ldots + \varepsilon^m\Phi_+^{(m)} \tag{3.3}$$

Inserting the latter into (3.2) and comparing the coefficients of like powers of ε, we obtain the following system of functional equations

$$\Phi_+^{(0)}(\alpha)\,L_1^*(\alpha)\,\alpha^{-1} = F_+(\alpha) + E_-(\alpha) \tag{3.4}$$

$$\Phi_+^{(i)}(\alpha)\;L_1^*(\alpha)\,\alpha^{-1} + \Phi_+^{(i-1)}(\alpha)\;L_2^*(\alpha)\,\alpha^{-1} = 0 \tag{3.5}$$

In view of the fact that $\phi\eta(x)$ has, in general (see Theorem 1.1), a singularity of the type $(1-x^2)^{-1/2}$ at the ends, we shall, taking into account (2.2), seek the solution of (2.3) in the class of functions satisfying the condition $\psi_{\pm}(\tau) \to \tau^{-1/2}$ as $\tau \to 0$. Applying the usual reasoning [4] we obtain,

$$\Phi_+^{(0)}(\alpha) = i\gamma_{\pm}(2\pi)^{-1/2}(\alpha \mp \beta)^{-1}[L_1^*(\alpha)\alpha^{-1}]_+^{-1}[\pm L_1^*(\pm\beta)\beta^{-1}]_-^{-1}$$

$$L_1^*(\alpha)\alpha^{-1} = [L_1^*(\alpha)\alpha^{-1}]_+ [L_1^*(\alpha)\alpha^{-1}]_- \tag{3.6}$$

as the solution of the functional equation (3.3). This, substituted into (3.5) yields after some elementary manipulations

$$\Phi_+^{(1)}(\alpha) = \gamma_{\pm}(2\pi)^{-1/2}C_+(\alpha)[L_1^*(\alpha)\alpha^{-1}]_+^{-1}[\pm L^*(\pm\beta)\beta^{-1}]_-^{-1}$$

$$C_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{1}{i(\tau\mp\beta)}\frac{L_2^*(\tau)}{L_1^*(\tau)}\frac{d\tau}{\tau-\alpha} \tag{3.7}$$

Writing now $(\tau^2 + B^2)^{-1/2}$ as

$$(\tau^2 + B^2)^{-1/2} = g_+(\tau) + g_-(\tau)$$

$$g_+(\tau) = i\pi^{-1}(\tau^2 + B^2)^{-1/2}\ln\{(iB)^{-1}[\tau + (\tau^2 + B^2)^{1/2}]\} \tag{3.8}$$

$$g_-(\tau) = -i\pi^{-1}(\tau^2 + B^2)^{-1/2}\ln\{(-iB)^{-1}[\tau + (\tau^2 + B^2)^{1/2}]\}$$

we can easily obtain an expression for $C_+(\alpha)$ by writing the integral entering (3.7) as two integrals, one of them containing only the poles of the integrand function corresponding to $\operatorname{Im}\tau > c$, and the other the poles corresponding to $\operatorname{Im}\tau < c$. Further, substituting (3.7) into (3.6) and solving the resulting functional equation, we find $\Phi_+^{(2)}(\alpha)$ etc. The subsequent terms of (3.3) are more difficult to obtain, since difficulties are encountered in calculating integrals of the type (3.7); in practice $\Phi_+^{(1)}(\alpha)$ is often found sufficiently accurate.

Let us now construct a solution of (3.1), based on the approximation described in Section 2. We shall write the function $L(u)u^{-1}$ in the form

$$L(u)u^{-1} = \exp[c_1 M(u)]L^*(u)u^{-1}, \qquad M(u) = \frac{1}{\sqrt{u^2+c^2}}\prod_{n=1}^{N}\frac{u^2+a_n^2}{u^2+b_n^2}$$

Eq. (3.1) can be written as

$$\Phi_+(\alpha)L^*(\alpha)\alpha^{-1}\exp[c_1 M(\alpha)] = F_+(\alpha) + E_-(\alpha) \tag{3.9}$$

Solution of this equation will be

$$\Phi_+(\alpha) = i(2\pi)^{-1/2}\gamma_{\pm}(\alpha\mp\beta)^{-1}[L^*(\alpha)\alpha^{-1}]_+^{-1}[\pm L^*(\beta)\beta^{-1}]_-^{-1}\times$$

$$\times\exp\{-c_1[M_+(\alpha + M_-(\pm\beta)]\} \tag{3.10}$$

$$M_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ib}^{\infty+ib}\frac{1}{\sqrt{u^2+c^2}}\prod_{n=1}^{N}\frac{u^2+a_n^2}{u^2+b_n^2}\frac{du}{u-\alpha}$$

Function $M_+(\alpha)$ is easily obtained, if we take into account (3.8) and the note concerning the computation of the integral entering (3.7). Although the formula (3.10) gives a closed solution of (3.9) we find, that, to obtain $\psi_{\pm}(\tau)$ we must perform a complicated contour integration, using approximate methods. The computation simplifies if, taking into account the properties of $M(u)$ given at the end of Section 2, we write (3.10) in the form

$$\Phi_+(\alpha) = i(2\pi)^{-1/2}\gamma_{\pm}(\alpha\mp\beta)^{-1}[L^*(\alpha)\alpha^{-1}]_+^{-1}[\pm L^*(\pm\beta)\beta^{-1}]_-^{-1}\times$$

$$\times\exp[-c_1M_-(\pm\beta)]\sum_{k=0}^{\infty}(-1)^k\frac{c_1^kM_+^k(\alpha)}{k!} \tag{3.11}$$

We easily see that (3.11) is analogous to (3.3) with one exception, – that in (3.11), all $\Phi_+^i(\alpha)$ are already defined.

4. Examples. We shall consider the problems on the interaction between a stiff tire on a cylindrical surface (Problem a)(*), and between a stiff bushing and the surface of a cylindrical cavity in an elastic space (Problem b). We shall assume that friction is absent within the area of elastic contact, and, that there is no load outside this area. Using operational methods we can reduce these problems to the determination of the contact pressure $q(z)$ for the following integral equation [1]

$$\int_{-1}^{1} q(a\tau) K\left(\frac{\tau - z}{\lambda}\right) d\tau = \pi\delta \qquad \left(\begin{array}{c} |z| \leqslant a,\ \lambda = Ra^{-1},\ \delta = \Delta\gamma a^{-1} \\ \Delta = {}^1/_2 E(1 - \nu^2)^{-1} \end{array}\right)$$

$$K(t) = \int_0^\infty \frac{L(u)}{u} \cos ut\, du \qquad \left(t = \frac{\tau - z}{\lambda}\right)$$

Here a denotes the semi-width of the tire (bushing), R is the radius of the cylinder (cavity), γ denotes the depth of penetration of the tire (bushing) into the surface of the cylinder (cavity) and $L(u)u^{-1}$ is defined by Formulas (2.3) or (2.4) of [1].

To solve the problems, we shall use $L(u)u^{-1}$ as given by (2.25), putting in the latter

$$\frac{L_1(u)}{u} = \frac{\sqrt{u^2 + B^2}(u^2 + D^2)}{(u^2 + C^2)(u^2 + E^2)}, \qquad \frac{L_2(u)}{u} = \frac{c_1 u^2 (u^2 + d^2)}{(u^2 + C^2)(u^2 + E^2)(u^2 + e^2)}$$

Using Formulas (3.6), (3.7) and (3.2) we obtain

$$\psi_\pm^{(0)}(t) = \delta A^{-1/2}[e^{-Bt}(\pi t)^{-1/2} + A^{-1/2}\operatorname{erf}\sqrt{Bt} - r_0 e^{-Dt}\operatorname{erf}\sqrt{(B-D)t}]$$

$$\psi_\pm^{(1)}(t) = c_1\delta A^{-1/2}\{-S_0(t) - r_1(D)J_0^+(D,t) + r_1(e)J_0^+(e,t) + r_3 e^{-Dt}\operatorname{erf}\sqrt{(B-D)t} +$$
$$+ r_2\int_0^t e^{-D\tau}\operatorname{erf}\sqrt{(B-D)\tau}\,K_0[B(t-\tau)]\,d\tau + r_1(-e)J_0^-(-e,t) -$$
$$- A_1 D r_0 J_1^-(-D,t) - r_0 A_1 D(B-D)^{-1/2}[S_1(t) + g_+(iD)\pi^{-1/2}t^{1/2}\exp(-Bt)]\} \qquad (4.1)$$

where

$$r_0 = \frac{(C-D)(E-D)}{D\sqrt{B-D}}, \quad r_1(x) = \frac{(d-x^2)(C+x)(E+x)}{2(e^2-D^2)(D+x)\sqrt{B+x}}, \qquad A_1 = \frac{d-D^2}{2(e^2-D^2)}$$

$$r_2 = \frac{1}{\pi}\left[\frac{A_1 B r_0}{2(B-D)} - \frac{2A_2 D^2 r_0}{D^2 - e^2} - A_1(C + E - 2D)\right], \qquad A_2 = \frac{D - e^2}{2(e^2 - D^2)}$$

$$r_3 = \frac{A_1(2D - B)r_0 g_+(iD)}{2(B-D)} - \frac{2A_2 D e r_0 g_+(ie)}{D^2 - e^2} - A_1(C + E - 2D)g_+(iD)$$

$$S_k(t) = \frac{1}{\pi^{3/2}}\int_0^t \tau^{k-1/2}e^{-B\tau}K_0[B(t-\tau)]\,d\tau, \qquad A = \frac{BD^2}{C^2E^2} \qquad (4.2)$$

$$J_k^\pm(x,t) = \frac{1}{\pi}\int_0^t e^{x\tau}\operatorname{erf}\sqrt{(B+x)\tau}\,K_0[B(t-\tau)]\,\tau^k\,d\tau \mp g_\pm(ix)\,t^k e^{xt}\operatorname{erf}\sqrt{(B+x)t}$$

In the above formulas $K_0(x)$ is the MacDonald function, while the functions $g_\pm(ix)$ and $\Psi_\pm(t)$ have the form

$$g_+(ix) = \pi^{-1}(B^2 - x^2)^{-1/2}\ln B^{-1}[x + (B^2 - x^2)^{1/2}] = g_-(-ix) \quad \psi_\pm(t) = \psi_\pm^{(0)}(t) + \psi_\pm^{(1)}(t)$$

Approximating the function $L(u)u^{-1}$ with an error not exceeding 5% over the whole interval of variation of $u \in [0, \infty)$ we find, that for the Problem a we have: $B = 1$, $D = 1.0354$, $C = 1.7321$, $E = 0.9640$, $c_1 = 0.4$, $d = -0.4$, $e = 1.2247$, while for the Problem b we have: $B = 1$, $D = 1.0354$, $C = 1.264$, $E = 0.9694$, $c_1 = -0.4$ and $d = e = 0$.

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*) This problem for a semi-infinite tire was considered in [10 and 11], and for the finite tire – in [12 and 13]. Complete solution was, however, not obtained.

The Table gives, for comparison purposes, some values of $\phi^*(0) = \phi(0)\,\delta^{-1}$, $\phi^*(0.5) = \phi(0.5)\,\delta^{-1}$ and of $f(1) = \lim\,[\sqrt{1-x^2}\,\phi(x)\,\delta^{-1}]$ as $x \to 1$ obtained for $\lambda = 2$ from Formulas (4.1), (4.2) and (2.2), (2.5) of the present paper, and from Formulas (1.14) to (1.18) and (2.12), (2.13) of [1].

TABLE

		$\varphi^*(0)$	$\varphi^*(0,5)$	$f(1)$
a	(1.14)—(1.18) [1]	1.170	1.241	0.923
	(2.12) [1]	1.223	1.304	0.898
	(4.1), (4.2), (2.2), (2.5)	1.222	1.308	0.910
b	(1.14)—(1.18) [1]	0.988	1.025	0.666
	(2.13) [1]	0.931	1.018	0.697
	(4.1), (4.2), (2.2), (2.5)	0.936	1.025	0.668

From these data we can infer, that the solution obtained in the present paper gives a good agreement at $\lambda = 2$, and the method implies that its accuracy will increase with decreasing λ. We can therefore conclude, that this solution and the solution given in [1], make possible the investigation of the given class of integral equations over the whole range of values of λ.

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